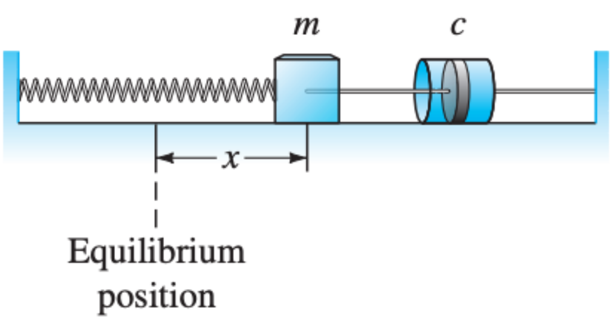
Lecture 16. Mechanical Vibrations Part 1

Mass-spring-dashpot system



- Restorative force $F_S = -kx$, where $k > 0$ is **spring constant** (Hooke's law).
- The dashpot provides force $F_R = -cv = -c\dfrac{dx}{dt}$, where $c > 0$ is **damping constant**.
- **External force** $F_E = F(t)$.
- The total force acting of the mass is $F = F_S + F_R + F_E$.
- Using Newton's law,

$$F = ma = m\frac{d^2x}{dt^2} = mx''$$

  we have the following second-order linear differential equation

$$mx'' + cx' + kx = F(t) \tag{1}$$

- If $c = 0$, we call the motion **undamped**. If $c > 0$, we call the motion **damped**.
- If $F(t) = 0$, we call the motion **free**. If $F(t) \neq 0$, we call the motion **forced**.

🤓 **An important note before we start analyzing the general cases:**

Rather than memorizing the various formulas given in the discussion below, it is better to practice a particular case to set up the differential equation and then solve it directly.

1. Free Undamped Motion ($c = 0$ and $F(t) = 0$)

Our general differential equation takes the simpler form

$$mx'' + kx = 0.$$

- It is convenient to define

$$\omega_0 = \sqrt{\frac{k}{m}}$$

- Then we can rewrite our equation in the form

$$x'' + \omega_0^2 x = 0$$

The char. eqn. is

$r^2 + w_0^2 = 0$

$\Rightarrow r^2 = -w_0^2$

$\Rightarrow r = \pm w_0 i = 0 \pm w_0 i$

$\Rightarrow x(t) = e^{0t}(A\cos w_0 t + B\sin w_0 t)$

- The general solution of this equation is

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

We write $x(t) = C\left(\frac{A}{C}\cos w_0 t + \frac{B}{C}\sin w_0 t\right)$

$= C(\cos\alpha\cos w_0 t + \sin\alpha\sin w_0 t)$, where $\cos\alpha = \frac{A}{C}$, $\sin\alpha = \frac{B}{C}$

Recall $\cos(a-b) = \cos a\cos b - \sin a\sin b$

$\Rightarrow x(t) = C\cos(w_0 t - \alpha)$ By writing $x(t)$ in this format, it is easier to anlylize it's property, includes period, amplitude, frenquency.

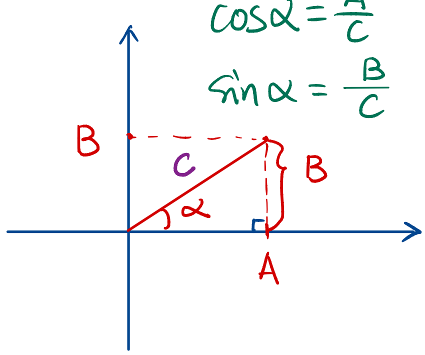
- **Question 1**. What is the values of C ?

$C = \sqrt{A^2 + B^2}$

- **Question 2**. What is the angle α ?

$\cos\alpha = \frac{A}{C}$

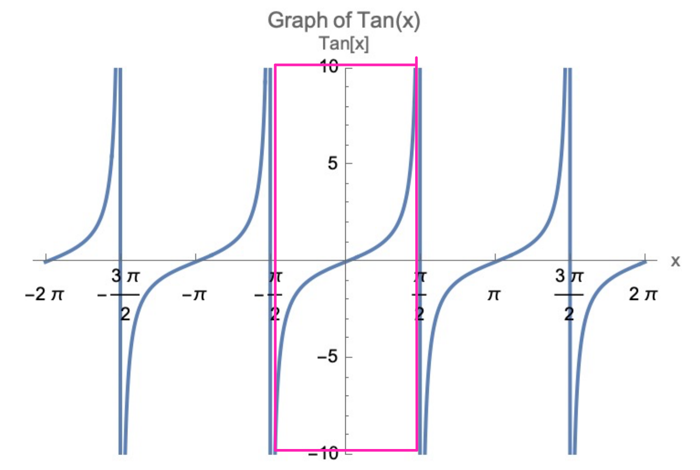
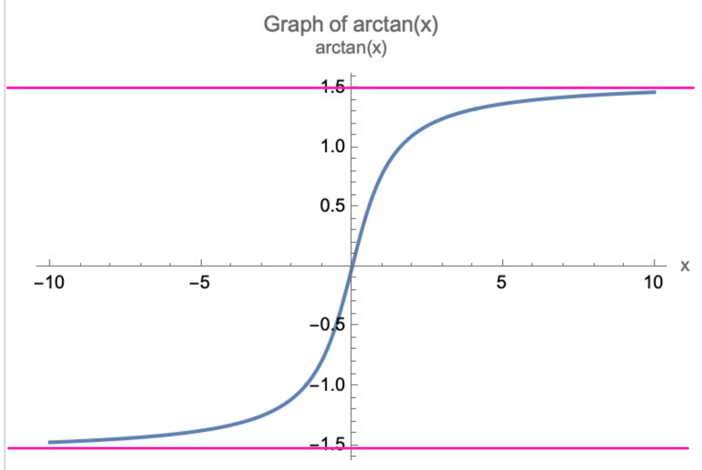
$\sin\alpha = \frac{B}{C}$



$\Rightarrow \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{B}{C}}{\frac{A}{C}} = \frac{B}{A}$

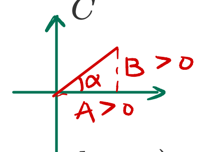
Recall $\tan^{-1}\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

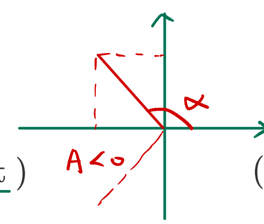
However, we want to express $\alpha \in [0, 2\pi)$

$\tan x$	$\arctan x$ $= \tan^{-1} x$
	

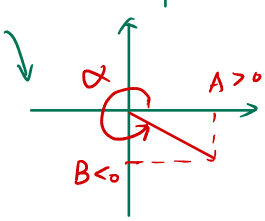
Remark:

- Although $\tan\alpha = \dfrac{B}{A}$, the angle α is not given by the principal branch of the inverse tangent function, which gives value only in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
- Instead, α is the angle between 0 and 2π such that $\sin\alpha = \dfrac{B}{C}, \cos\alpha = \dfrac{A}{C}$. where either A or B might be negative.
- Thus





$$\alpha = \begin{cases} \tan^{-1}(B/A) & \text{if } A > 0, B > 0 (\text{ first quadrant }) \\ \pi + \tan^{-1}(B/A) & \text{if } A < 0 (\text{ second or third quadrant }) \\ 2\pi + \tan^{-1}(B/A) & \text{if } A > 0, B < 0 \text{ (fourth quadrant)} \end{cases} \tag{2}$$



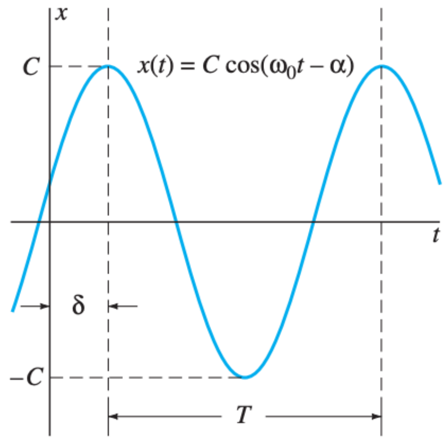
where $\tan^{-1}(B/A)$ is the angle in $(-\pi/2, \pi/2)$ given by a calculator or computer.

- So we have

$$x(t) = C\cos(\omega_0 t - \alpha)$$

where ω, C and α are obtained as above.

- We call such motion **simple harmonic motion.** A typical graph of such motion is as



- To summarize , it has

Name	Symbol	Quick note
Amplitude	C	$C = \sqrt{A^2 + B^2}$, where $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$ is the solution for the equation $x'' + \omega_0^2 x = 0$.
Circular frequency	ω_0	$\omega_0 = \sqrt{\dfrac{k}{m}}$
Phase angle	α	Obtained by formula (2) above
Period	$T = \dfrac{2\pi}{\omega_0}$	Time required for the system to complete one full oscillation
Frequency	$\nu = \dfrac{1}{T} = \dfrac{\omega_0}{2\pi}$ (In Hz)	It measures the number of complete cycles per second.

Example 1

- A body with mass $m = 0.5$ kilogram (kg) is attached to the end of a spring that is stretched 2 meters (m) by a force of 100 newtons (N).
- $x(0) = 1$ $\quad$ $V(0) = x'(0) = -5$
  It is set in motion with initial position $x_0 = 1$ (m) and initial velocity $v = -5$ (m/s). (Note that these initial conditions indicate that the body is displaced to the right and is moving to the left at time $t = 0$.)
- Find the position function of the body in the form $C\cos(\omega_0 t - \alpha)$ as well as the amplitude, frequency and period of its motion.

ANS: Find k: $\quad F = -kx \Rightarrow 100 = 2k \Rightarrow k = 50\,N/m$

Then we have

$$0.5\,x'' + 50x = 0, \quad x(0) = 1, \quad x'(0) = -5.$$

$$\Rightarrow \quad x'' + 100x = 0 \quad (x'' + w_0^2 x = 0, \quad w_0 = 10)$$

The char. eq is $r^2 + 100 = 0 \Rightarrow r = \pm 10i$

The general solution is $x(t) = A\cos 10t + B\sin 10t$

where A and B are constants.

As $x(0) = 1$, $x(0) = A = 1$,

As $x'(t) = -10A\sin 10t + 10B\cos 10t$. $\quad x'(0) = -5$

We know $x'(0) = 10B = -5 \Rightarrow B = -\frac{1}{2}$

Thus $x(t) = \cos 10t - \frac{1}{2}\sin 10t$

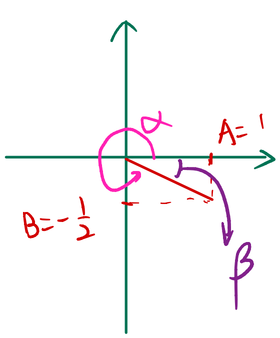
The amplitude of the motion is

$$C = \sqrt{A^2 + B^2} = \sqrt{1^2 + (-\tfrac{1}{2})^2} = \sqrt{1 + \tfrac{1}{4}} = \frac{\sqrt{5}}{2}\ m$$

Thus we have

$$x(t) = \cos 10t - \frac{1}{2}\sin 10t$$

$$= \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\cos 10t - \frac{1}{\sqrt{5}}\sin 10t\right)$$



By the graph, we know α lies in the 4th quadrant.

$(\frac{3\pi}{2} < \alpha < 2\pi)$

We treat β as an angle in between 0 to $\frac{\pi}{2}$.

$$\alpha = 2\pi - \underset{\substack{\uparrow \\ \text{assuming } \beta \in (0, \frac{\pi}{2})}}{\underline{\beta}} = 2\pi - \tan^{-1}\frac{\frac{1}{2}}{1} = 2\pi - \tan^{-1}\frac{1}{2} \approx 5.8195 \text{ rad}$$

You can also use the formula on page 3 by identifying α is in the 4th quadrant.

2. Free Damped Motion ($c > 0$ and $F(t) = 0$)

In this case, we consider

$$mx'' + cx' + kx = 0 \Rightarrow x'' + \overset{2p}{\cancel{\tfrac{c}{m}}}\, x' + \overset{w_0^2}{\cancel{\tfrac{k}{m}}}\, x = 0$$

Let $\omega_0 = \sqrt{k/m}$ and $p = \dfrac{c}{2m} > 0$. We have

$$x'' + 2px' + \omega_0^2 x = 0$$

The characteristic equation

$$r^2 + 2pr + \omega_0^2 = 0 \Rightarrow r_{1,2} = \frac{-2p \pm \sqrt{4p^2 - 4w_0^2}}{2} \tag{3}$$

has roots

$$r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$$

→ Recall there are 3 cases

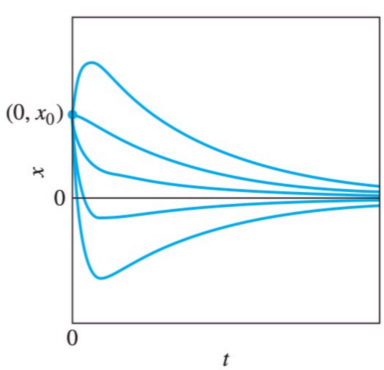
1. distinct real roots
2. Repeated roots
3. Complex conjugate roots

Note

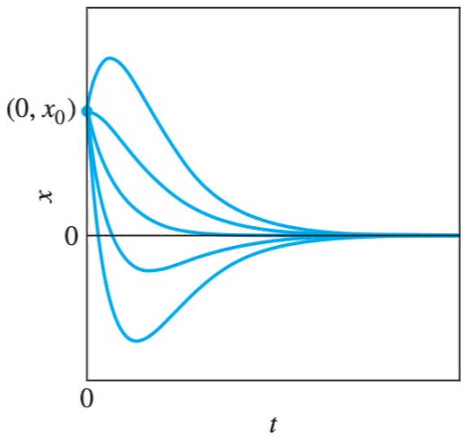
$$p^2 - \omega_0^2 = \frac{c^2 - 4km}{4m^2}$$

We have the following three cases.

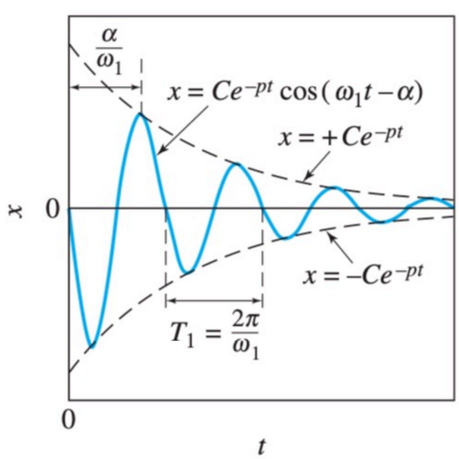
Case 1. Overdamped ($c^2 > 4km$, two distinct real roots)

Figure	Analysis
 **FIGURE 3.4.7.** Overdamped motion: $x(t) = c_1e^{r_1t} + c_2e^{r_2t}$ with $r_1 < 0$ and $r_2 < 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.	Eq(3) gives two distinct real roots r_1 and r_2(both < 0). The position function $$x(t) = c_1e^{r_1t} + c_2e^{r_2t}$$ Note $$\lim_{t\to\infty} x(t) = 0$$ (The object will go to the equilibrum position without any ocillations.)

Case 2. Critically damped ($c^2 = 4km$, repeated real roots)

Figure	Analysis
 **FIGURE 3.4.8.** Critically damped motion: $x(t) = (c_1 + c_2 t)e^{-pt}$ with $p > 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.	Eq(3) has roots $r_1 = r_2 = -p$. The general solution for the position function. $$x(t) = e^{-pt}\left(c_1 + c_2 t\right)$$ and $$\lim_{t\to\infty} x(t) = 0$$ The resistance of the dashpot is just enough to damp out any oscillations.

Case 3. Underdamped ($c^2 < 4km$, two complex roots)

Figure	Analysis
 **FIGURE 3.4.9.** Underdamped oscillations: $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$.	Eq(3) has roots $r_{1,2} = -p \pm i\sqrt{{\omega_0}^2 - p^2} = -p \pm \omega_1 t$, where $w_1 = \sqrt{\omega_0^2 - p^2}$ The general solution for the position function $x(t) = \underbrace{e^{-pt}}_{\to 0 \text{ as } t\to\infty}\underbrace{\left(A\cos w_1 t + B\sin w_1 t\right)}_{\text{bounded function}}$ $\lim_{t\to\infty} x(t) = 0$

Example 2.

Suppose that the mass in a mass–spring–dashpot system with $m = 6, c = 7$, and $k = 2$ is set in motion with $x(0) = 0$ and $x(0)' = 2$.

(a) Find the position function $x(t)$.

(b) Find how far the mass moves to the right before starting back toward the origin.

ANS: (a) We have

$$6x'' + 7x' + 2x = 0, \quad x(0) = 0, \quad x'(0) = 2.$$

The char. eqn $6r^2 + 7r + 2 = 0$

$$\Rightarrow r_{1,2} = \frac{-7 \pm \sqrt{49-48}}{12} = \frac{-7 \pm 1}{12} = -\frac{2}{3}, -\frac{1}{2}$$

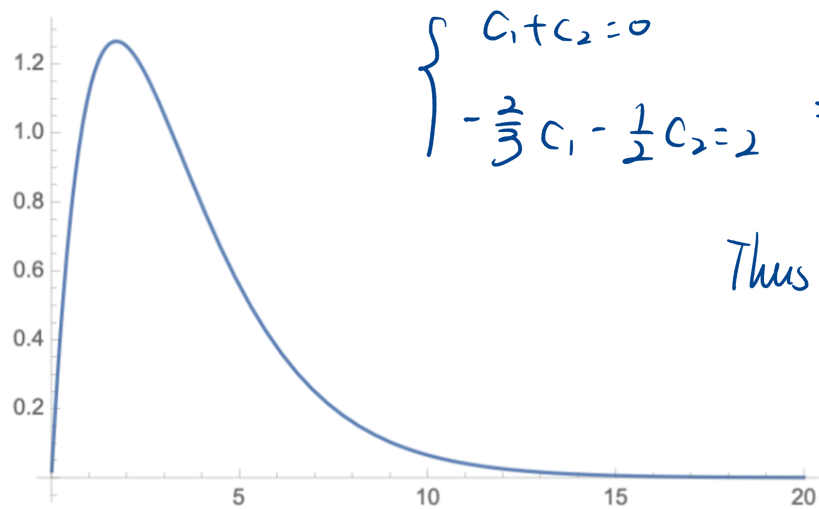
Thus $x(t) = C_1 e^{-\frac{2}{3}t} + C_2 e^{-\frac{1}{2}t}$.

$x(0) = 0$, $C_1 + C_2 = 0$ Since $x'(0) = 2$.

$$x'(t) = -\frac{2}{3} C_1 e^{-\frac{2}{3}t} - \frac{1}{2} C_2 e^{-\frac{1}{2}t}$$

$$x'(0) = -\frac{2}{3} C_1 - \frac{1}{2} C_2 = 2$$

$$\begin{cases} C_1 + C_2 = 0 \\ -\frac{2}{3} C_1 - \frac{1}{2} C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = -12 \\ C_2 = 12. \end{cases}$$



Thus $x(t) = -12 e^{-\frac{2}{3}t} + 12 e^{-\frac{1}{2}t}$

(b) The mass starts to move back when $x'(t) = 0$

Let $x(t) = 8e^{-\frac{2}{3}t} - 6e^{-\frac{t}{2}} = 0 \Rightarrow \overset{4}{\cancel{8}}\, e^{-\frac{2}{3}t} = \overset{3}{\cancel{6}}\, e^{-\frac{t}{2}}$

$$\Rightarrow \frac{4}{3} = \frac{e^{-\frac{t}{2}}}{e^{-\frac{2}{3}t}} = e^{+\frac{2}{3}t - \frac{t}{2}} = e^{\frac{4-3}{6}t} = e^{\frac{t}{6}}$$

$$\Rightarrow \ln\frac{4}{3} = \frac{t}{6}$$

$$\Rightarrow t = \ln\frac{4}{3} \approx 1.72609\,s$$

Thus $x(6\ln\frac{4}{3}) = \frac{81}{64} \approx 1.26563\,m$

Exercise 3. For the differential equation

$$s'' + bs' + 9s = 0,$$

find the values of b that make the general solution overdamped, underdamped, or critically damped.

Solution.

The corresponding characteristic equation is

$$r^2 + br + 9 = 0$$

From the previous discussion, we know the general solution is overdamped when the solution for r has two distinct roots. It is underdamped if the solution for r is a pair of complex numbers. It is critically damped if the solution for r is repeated. Also we know b represents the damping constant, so $b > 0$.

Therefore,

- the system is overdamped when $\Delta = b^2 - 4 \times 9 > 0 \implies b^2 > 36$. Combing the fact that $b > 0$ we know $b > 6$.
- the system is critically damped when $\Delta = b^2 - 4 \times 9 = 0 \implies b = \pm 6$. Combing the fact that $b > 0$ we know $b = 6$.
- the system is underdamped when $\Delta = b^2 - 4 \times 9 < 0 \implies b^2 < 36$. Combing the fact that $b > 0$ we know $0 < b < 6$.

Exercise 4.

(1) Using a trig identity, write $x(t) = -\cos(9t) + 5\sin(9t)$ using only one cosine function.

(2) Using a trig identity, write $x(t) = \cos(9t) + 5\sin(9t)$ using only one cosine function.

(3) Using a trig identity, write $x(t) = e^{-3t}(-\cos(9t) + 5\sin(9t))$ using only one cosine function in your answer.

Solution.

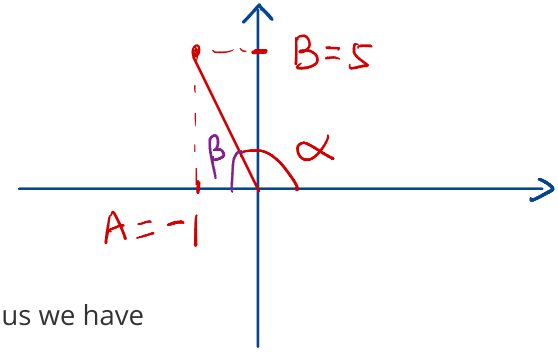
(1) Let $x(t) = -\cos(9t) + 5\sin(9t) = A\cos(9t) + B\sin(9t) = C\cos(\theta - \alpha)$, where $A = -1$ and $B = 5$.

From the discussion in our lecture notes, we know α is in the second quadrant and

$C = \sqrt{A^2 + B^2} = \sqrt{(-1)^2 + 5^2} = \sqrt{1 + 25} = \sqrt{26}$

You can either apply the formula given in Eq (3) or draw the triangle and compute the value of α.

- Applying the formula, we know $\alpha = \pi + \arctan(-5) = \pi - \arctan(5)$.
- Or we can draw the following triangle with sides A, B, and C, we have $\alpha = \pi - \arctan(5)$, which is the same as applying the formula.



Consider $\beta \in (0, \frac{\pi}{2})$, then

$\tan \beta = \left|\frac{5}{1}\right| = 5 \Rightarrow \beta = \arctan 5$

Thus $\alpha = \pi - \beta = \pi - \arctan 5$

Thus we have

$x(t) = -\cos(9t) + 5\sin(9t) = \sqrt{26}\cos(9t - \pi + \arctan(5))$.

(2) The steps for solving this problem is similar to previous one. Note this time α is in the first quadrant. So we have

$x(t) = -\cos(9t) + 5\sin(9t) = \sqrt{26}\cos(9t - \arctan(5))$

(3) Note the steps for solving this case is again similar to the question (1). The difference is that we need to multiply e^{-5t} everywhere in $x(t)$. So we have

$x(t) = e^{-3t}(-\cos(9t) + 5\sin(9t)) = \sqrt{26}e^{-3t}\cos(9t - \pi + \arctan(5))$

Exercise 5. If the differential equation

$$m\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 4x = 0$$

is overdamped, the range of values for m is?

Solution.

The corresponding characteristic equation is

$$mr^2 + 8r + 4 = 0$$

The system is overdamped if it has two distinct solutions for r. That is when $\Delta = 8^2 - 16m > 0$

Also as m is representing the mass of the object, $m > 0$. So we have $0 < m < 4$.